E0 232: PROBABILITY AND STATISTICS

PROBLEM SHEET 1
TUTORIAL ON 25^{TH} AUGUST, 2014
VENUE: CSA - 252 (5.00 PM ONWARDS)

1. **Probability space and axioms**
   1.1. [**Ross 1.3**] A coin is to be tossed until a head appears twice in a row. What is the sample space for this experiment (give explicit description)? If the coin lands on head with probability p, what is the probability that it will be tossed exactly four times?
   1.2. [**Rohatgi 1.3.9**] Prove the principle of inclusion-exclusion.
   Let $A_1, A_2, \ldots, A_n \in \mathcal{S}$ (event space). Then

$$P\left(\bigcup_{k=1}^{n} A_i\right) = \sum_{k=1}^{n} P(A_k) - \sum_{k_1<k_2} P(A_{k_1} \cap A_{k_2}) + \sum_{k_1<k_2<k_3} P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) \ldots + (-1)^{n+1} P\left(\bigcap_{k=1}^{n} A_i\right)$$

   1.3. [**Rohatgi 1.5.1**] Let A and B be two events such that $P(A) = p_1 > 0$, $P(B) = p_2 > 0$, and $p_1 + p_2 > 1$- Show that $P(B|A) \geqslant \left(1 - \frac{1-p_2}{p_1}\right)$.
   1.4. [**Rohatgi 1.6.9**] Let $(\Omega, \mathcal{S}, P)$ be a probability space. Let $A, B, C \in \mathcal{S}$ with $P(B)$ and $P(C) > 0$. If B and C are independent, show that

$$P(A|B) = P(A|B \cap C)P(C) + P(A|B \cap C^C)P(C^C).$$

   Conversely, if this relation holds, $P(A|B \cap C) \neq P(A|B)$, and $P(A) > 0$, then show that B and C are independent
   1.5. [**Hajek 1.4**] (i) Show that an event E is independent of itself if and only if either $P(E) = 0$ or $P(E) = 1$.
   (ii) Events A and B have probabilities $P(A) = 0.6$ and $P(B) = 0.8$. Could the events be independent? Could they be mutually exclusive?

These problems have been taken from:

- Chapters 1,2 of *An Introduction to Probability and Statistics* by Rohatgi and Saleh, 2nd ed.
- Chapter 1 of *Introduction to Probability Model* by Sheldon Ross, 10th edition.
- Chapter 1 of *An Exploration of Random Processes for Engineers* by Bruce Hajek, 2014.

Some of the problems have been modified slightly. The problems given here are not straightforward, and hence, solving other problems from suggested textbooks before trying these may be useful. Solutions of these problems will be discussed during the tutorial session.

1.6. **[Ross 1.28]** If the occurrence of B makes A more likely, does the occurrence of A make B more likely?

2. **Calculating probability by reasoning**

2.1. **[Rohatgi 1.3.7]** Two points are chosen at random on a line of unit length. Find the probability that each of the three line segments so formed will have a length $> \frac{1}{4}$.

2.2. **[Ross 1.14]** The probability of winning on a single toss of the dice is p. A starts, and if he fails, he passes the dice to B, who then attempts to win on her toss. They continue tossing the dice back and forth until one of them wins. What are their respective probabilities of winning?

2.3. **[Rohatgi 1.4.15]** Consider a town with N people. A person tells a rumor to a second person, who in turn repeats it to a third person, and so on. Suppose that at each stage the recipient of the rumor is chosen at random from the remaining $(N-1)$ people. What is the probability that the rumor will be repeated n times:

(i) Without being repeated to any person?

(ii) Without being repeated to the originator?

2.4. **[Ross 1.12]** Let E and F be mutually exclusive events in the sample space of an experiment. Suppose that the experiment is repeated until either event E or event F occurs. Show that the probability that event E occurs before event F is $P(E)/[P(E)+P(F)]$.

Hint: Argue that the probability that the original experiment is performed n times and E appears on the n^{th} time with probability $P(E) \times (1-P(E)-P(F))^{n-1}$, $n = 1, 2, \ldots$. Add these probabilities to get the desired answer.

2.5. **[Rohatgi 1.5.14]** A diagnostic test for a certain disease is 95 percent accurate, in that if a person has the disease, it will detect it with a probability of 0.95, and if a person does not have the disease, it will give a negative result with a probability of 0.95. Suppose that only 0.5 percent of the population has the disease in question. A person is chosen at random from this population. The test indicates that this person has the disease. What is the (conditional) probability that he or she does have the disease?

2.6. **[Ross 1.45]** An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn c additional balls of the same color are put in with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is $b/(b+r+c)$.

2.7. [**Hajek 1.2**] Suppose there is an election with two candidates and six ballots turned in, such that four of the ballots are for the winning candidate and two of the ballots are for the other candidate. The ballots are opened and counted one at a time, in random order, with all orders equally likely. Find the probability that from the time the first ballot is counted until all the ballots are counted, the winning candidate has the majority of the ballots counted. ("Majority" means there are strictly more votes for the winning candidate than for the other candidate.)

3. **Random variables**

3.1. [**Rohatgi 2.2.1, 2.3.1**] Let X be the number of heads in three tosses of a coin. What is Ω? What are the values that X assigns to points of Ω? What are the events $\{X \leq 2.75\}$, $\{0.5 \leq X \leq 1.72\}$? Write the distribution function of X assuming that p is probability of getting head.

3.2. [**Rohatgi 2.2.3**] Let X be an RV (random variable) on the sample space $(\Omega, \mathcal{F})$. Is $|X|$ also an RV? If X is an RV that takes only nonnegative values, is $\sqrt{X}$ also an RV?
Hint: Use the definition of RV in the following form. For any real x, the RV X satisfies $X^{-1}\big((-\infty, x]\big) \in \mathcal{F}$. Using this and the properties of σ-field, show that the same holds for $|X|$ and $\sqrt{X}$.