E0 232: PROBABILITY AND STATISTICS

PROBLEM SHEET 3
TUTORIAL ON 18^{TH} SEPT, 2014 (5.30 PM ONWARDS)
VENUE: WILL BE ANNOUNCED

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(1) **[Rohatgi 3.2.9,3.2.10]** $\lambda > 0$ is a constant, X is the Poisson RV with PMF $P\{X = x\} = \frac{\lambda^x}{x!}e^{-\lambda}$, $x = 0, 1, 2, \ldots$. Show / compute the following:

(a) mean / 1^{st} moment, $\mu = E[X] = \lambda$
(b) 2^{nd} moment, $E[X^2] = \lambda + \lambda^2$
(c) 3^{rd} moment, $E[X^3] = \lambda + 3\lambda^2 + \lambda^3$
(d) 4^{th} moment, $E[X^4] = \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4$
(e) variance / 2^{nd} central moment, $Var(X) = \mu_2 = E[(X - \mu)^2] = \lambda$
(f) 3^{rd} central moment, $\mu_3 = E[(X - \mu)^3] = \lambda$
(g) 4^{th} central moment, $\mu_4 = E[(X - \mu)^4] = \lambda + 3\lambda^2$
(h) coefficient of skewness, $\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} = ?$
(i) kurtosis, $\alpha_4 = \frac{\mu_4}{(\mu_2)^2} = ?$

(2) **[Rohatgi 3.2.2]** From a box containing N identical tickets numbered 1 to N, n tickets are drawn with replacement. Let X be the largest number drawn. Find $E[X]$.

(3) **[Rohatgi 3.2.14]** Compute $E[X]$, $Var(X)$, and $E[X^n]$ ($n > 0$ integer) for the following PMF, where for $0 < p < 1$,
$P(X = x) = p(1 - p)^{x-1}, x = 1, 2, \ldots,$ and zero elsewhere.

(4) **[Ross 2.51]** A coin, having probability p of landing heads, is flipped until a head appears for the r^{th} time. Let N denote the number of flips required. Calculate $E[N]$.
Hint: There is an easy way of doing this. It involves writing N as the sum of r geometric RVs. Otherwise, recall one problem from previous problem sheet.

These problems have been taken from:

- Chapters 2 and 3 of *An Introduction to Probability and Statistics* by Rohatgi and Saleh, second edition.
- Chapter 2 of *Introduction to Probability Model* by Sheldon Ross, tenth edition.
- Chapter 1 of *An Exploration of Random Processes for Engineers* by Bruce Hajek, 2014.

Some of the problems have been modified slightly. The problems given here are not straightforward, and hence, solving other problems from suggested textbooks before trying these may be useful. Solutions of these problems will be discussed during the tutorial session.

(5) **[Ross 2.46]** If X is a nonnegative integer valued random variable, show that

$$E[X] = \sum_{n=1}^{\infty} P\{X \geq n\} = \sum_{n=0}^{\infty} P\{X > n\}$$

Hint: Define the sequence of RVs I_n, $n \geq 1$, as $I_n = 1$ if $X \geq n$, 0 otherwise. Now express X in terms of the I_n.

(6) **[Hajek 1.19]** Suppose the length L and width W of a rectangle are independent and each uniformly distributed over the interval $[0, 1]$. Let $C = 2L + 2W$ (the length of the perimeter) and $A = LW$ (the area).
(a) Find the means, variances, and probability densities of C and A.
(b) Solve the above problem when L and W are exponentially distributed with parameter $\lambda > 0$.

(7) **[Hajek 1.29]** Let the random variables X and Y be jointly uniformly distributed on the region $\{0 \leq u \leq 1, 0 \leq v \leq 1\} \cup \{-1 \leq u < 0, -1 \leq v < 0\}$.
(a) Determine the joint pdf f_{XY}.
(b) Find f_X, the marginal pdf of X.
(c) Find the conditional pdf of Y given that $X = a$, for $-1 \leq a \leq 1$.
(d) Find $E[Y|X = a]$ for $|a| \leq 1$.
(e) Are X and Y independent?
(f) What is the pdf of $Z = X + Y$?
(g) Find $\rho_{XY} = \dfrac{E[(X - EX)(Y - EY)]}{\sqrt{Var(X)Var(Y)}}$.
(ρ_{XY} is called the correlation coefficient of X and Y.)
You need not try (c) and (d) if conditional distribution and conditional expectation are not covered in class.

SOLUTIONS

(1) Only $E[X^2]$ and $Var(X)$ are computed here. Hopefully, this will show the tricks involved.

$$\begin{aligned}
E[X^2] &= \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{(x-1)!} \\
&= \sum_{x=1}^{\infty} (x-1) \frac{\lambda^x e^{-\lambda}}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!} \\
&= \sum_{x=2}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-1)!} \\
&= \lambda^2 \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} + \lambda \sum_{z=0}^{\infty} \frac{\lambda^z e^{-\lambda}}{z!}
\end{aligned}$$

where we substitute $y = x - 2$ in first summation and $z = x - 1$ in second. Each summation in last step is 1 as it is sum of pmf's of Poisson distribution. So, $E[X^2] = \lambda^2 + \lambda$.

Now,

$$Var(X) = E[(X-\mu)^2] = E[X^2] - 2\mu E[X] + \mu^2$$

Note $\mu = E[X] = \lambda$ and $E[X^2]$ is computed above.

(2) Let X_i be ticket number of card chosen in i^{th} draw. So, $X = \max\{X_1, \ldots, X_n\}$. Observe that:

i. $X_1, \ldots, X_n$ are independent and each is uniformly distributed over $\{1, \ldots, N\}$.

ii. the event $\{X \leq k\}$ is same as the event $\{X_i \leq k\}$ for all i.

So, for any $k = 1, \ldots, N$,

$$\begin{aligned}
P(X \leq k) &= P\left(\bigcap_{i=1}^{n} \{X_i \leq k\}\right) \\
&= \prod_{i=1}^{n} P(X_i \leq k) \qquad \text{(due to independence)} \\
&= \left(\frac{k}{N}\right)^n.
\end{aligned}$$

Thus,

$$\begin{aligned}
E[X] &= \sum_{k=1}^{N} kP(X=k) \\
&= \sum_{k=1}^{N} k\big(P(X \le k) - P(X \le k-1)\big) \\
&= \frac{1}{N^n} \sum_{k=1}^{N} k^{n+1} - k(k-1)^n \\
&= N - \sum_{k=1}^{N} \left(\frac{k-1}{N}\right)^n
\end{aligned}$$

(3)

$$\begin{aligned}
E[X] &= \sum_{x=1}^{\infty} xp(1-p)^{x-1} \\
&= -p \sum_{x=1}^{\infty} \frac{d}{dp}(1-p)^x \\
&= -p\frac{d}{dp}\left(\sum_{x=1}^{\infty}(1-p)^x\right) \\
&= -p\frac{d}{dp}\left(\frac{1-p}{1-(1-p)}\right) = \frac{1}{p}
\end{aligned}$$

Similarly, you can compute $E[X^2] = \frac{2-p}{p^2}$ be rearranging terms and eventually using the fact

$$\frac{d^2}{dp^2}(1-p)^x = x(x-1)(1-p)^{x-2}$$

Hence, you can compute $Var(X)$.

To compute $E[X^n]$, you need to know moment generating function which will be done in class.

(4) Let the coin is flipped until head appears, and the number of tosses required be X_1. Obviously, X_1 is a geometric random variable with $E[X_1] = \frac{1}{p}$. Try this experiment r times, to get r such random variables $X_1, \ldots, X_r$.

Observe that $N = \sum_{i=1}^{r} X_i$. So $E[N] = \sum_{i=1}^{r} E[X_i] = \dfrac{n}{p}$.

(5) Observe $E[X] = \sum_{i=1}^{\infty} iP(X = i) = \sum_{i=1}^{\infty}\sum_{n=1}^{i} P(X = i)$. So we try to compute the sum of all elements in following table.

	$n \rightarrow$			
i	$P(X=1)$			
$\downarrow$	$P(X=2)$	$P(X=2)$		
	$P(X=3)$	$P(X=3)$	$P(X=3)$	
	$P(X=4)$	$P(X=4)$	$P(X=4)$	$P(X=4)$
	$\vdots$	$\vdots$	$\vdots$	

The form $E[X] = \sum_{i=1}^{\infty}\sum_{n=1}^{i} P(X = i)$ first adds every row, and then adds the sums of each row. Alternatively, one can sum each column, and then add sums of each column, which gives the following form

$$\begin{aligned}
E[X] &= \sum_{n=1}^{\infty}\sum_{i=n}^{\infty} P(X=i) \\
&= \sum_{n=1}^{\infty} P(X \geq n) && \text{(first form given in question)} \\
&= \sum_{n=0}^{\infty} P(X \geq n+1) \\
&= \sum_{n=0}^{\infty} P(X > n) && \text{(second form given in question)}
\end{aligned}$$

(6) Recall the following result done in class. If L and W are random variables with joint pdf f_{LW}, and if $A = LW$ and $B = L + W$, then

$$\begin{aligned}
f_A(a) &= \int_{-\infty}^{\infty} f_{LW}\left(x, \frac{a}{x}\right)\frac{1}{|x|}dx \\
f_B(b) &= \int_{-\infty}^{\infty} f_{LW}\left(x, b-x\right)dx
\end{aligned}$$

Note that L, W independant and uniform. So, $f_{LW}(l,w) = f_L(l)f_W(w) = 1$ if $l, w \in [0,1]$, and 0 otherwise.

Using this, one can derive

$$f_A(a) = \begin{cases} -\log a & \text{if } a \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

which gives pdf of area. Also

$$f_B(b) = \begin{cases} b & \text{if } b \in (0,1] \\ 2-b & \text{if } b \in (1,2) \\ 0 & \text{otherwise.} \end{cases}$$

From here, one can compute cdf of $F_B(b)$ Note that $C = 2L + 2W = 2B$, and verify that $F_C(c) = P(C \leq c) = P(B \leq c/2) = F_B(c/2)$. Using these, you can compute pdf

$$f_C(c) = \begin{cases} 0.25c & \text{if } c \in (0,2] \\ 1 - 0.25c & \text{if } b \in (2,4) \\ 0 & \text{otherwise.} \end{cases}$$

For the case, where L, W are exponentially distributed, it may be difficult to compute f_A. However, the same procedure as above gives

$$f_B(b) = \begin{cases} \lambda^2 b e^{-\lambda b} & \text{if } b > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and so,

$$f_C(c) = \begin{cases} 0.5(\lambda + 0.5c\lambda - 0.5)e^{-0.5c\lambda} & \text{if } c > 0 \\ 0 & \text{otherwise,} \end{cases}$$

(7) (a) Joint pdf of X and Y

$$f_{XY}(x,y) = \begin{cases} 0.5 & \text{if } x,y \in [-1,0] \\ 0.5 & \text{if } x,y \in [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

(b) If $x \in [0,1]$,

$$f_X(x) = \int_0^1 0.5 dy = 0.5$$

and if $x \in [-1,0]$

$$f_X(x) = \int_{-1}^0 0.5 dy = 0.5$$

So marginal pdf of X

$$f_X(x) = \begin{cases} 0.5 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise,} \end{cases}$$

Note f_Y is also similar.

(c) If $a \in [0,1]$

$$f_{Y|X}(y|a) = \frac{f_{XY}(a,y)}{f_X(a)} \qquad = \begin{cases} 1 & \text{if } y \in [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

Similarly, compute for $a \in [-1,0]$.

(d) If $a \in [0, 1]$

$$E[Y = y|a] = \int_0^1 y f_{Y|X}(y|a) dy = 0.5$$

Similarly, for $a \in [-1, 0]$,

$$E[Y = y|a] = \int_{-1}^0 y f_{Y|X}(y|a) dy = -0.5$$

(e) $f_{XY}(x, y) \neq f_X(x) f_Y(y)$.
So X and Y are not independant.

(f) Use formula for fing f_{X+Y}.