E0 232: PROBABILITY AND STATISTICS

PROBLEM SHEET 4

(1) A random vector $X = (X_1, ..., X_{n-1})$ is said to be a categorical random vector if

$$P(X = (x_1, ..., x_{n-1})) =$$
$$\begin{cases} p_k & \textbf{if } x_k = 1 \textbf{ and } x_i = 0 \quad \forall 1 \le i \le n-1, i \ne k, \\ 1 - \sum_{k=1}^{n-1} p_k & \textbf{if } x_k = 0 \ \ \forall 1 \le k \le n-1 \end{cases}$$

  (a) Find the MGF of X.
  (b) Find EX and the covariance matrix of X.

(2) Let $X^{(1)}, ..., X^{(m)}$ be identical and independent categorical random vectors as defined in the previous question and $Y = \sum_{j=1}^{m} X^{(j)}$.
  (a) Find the pmf of Y.
  (b) Find the MGF of Y.
  (c) Find EY and the covariance matrix for Y.

(3) Let $X_1, ..., X_m$ be m independent Poisson random variables such that $X_i = P(\lambda_i)$. Let $Y = \sum_{i=1}^{m} X_i$. Show that Y is also Poisson. Find the parameters of the corresponding distribution.

(4) [**Rohatgi 5.2.2**] Let $X_1, ..., X_m$ be m independent Poisson random variables such that $X_i \sim P(\lambda_i), i = 1, ..., n$. Show that the conditional distribution of $X_1, ..., X_{m-1}$ given $\sum_{i=1}^{m} X_i = t$ is multinomial. Also, find the parameters of the multinomial distribution.

(5) Let $X_1, ..., X_m$ be m independent binomial random variables such that $X_i = b(n_i, p)$. Let $Y = \sum_{i=1}^{m} X_i$. Show that Y is also binomial. Find the parameters of the corresponding distribution.

(6) [**Rohatgi 5.2.4**] A box contains N identical balls numbered 1 through N. Of these balls, n are drawn with replacement. Let $X_1, ..., X_n$ denote the numbers of the n balls drawn. Let $S_n = \sum_{i=1}^{n} X_i$. Find var(S_n).

These problems have been taken from:

- Chapter 5 of *An Introduction to Probability and Statistics* by Rohatgi and Saleh, second edition.
- Chapter 6 and 7 of *Probability, Random Variables, and Stochastic Processes* By Athanasios Papoulis, S. Unnikrishna Pillai

Some of the problems have been modified slightly. The problems given here are not straightforward, and hence, solving other problems from suggested textbooks before trying these may be useful. Solutions of these problems will be discussed during the tutorial session.

(7) [**Papoulis 6-49**] Let X and Y be independent random variables distributed as $\mathcal{N}(0, \sigma^2)$. If $Z = X - Y$, then show that $\mathrm{E}Z = \frac{2\sigma}{\sqrt{\pi}}$ and $\mathrm{var}(Z) = 2\sigma^2(1 - \frac{2}{\pi})$.

(8) [**Rohatgi Theorem 5.4.7**] Let $\mathbf{X} = (X_1, ..., X_n)$ be a random vector. Show that X has an n-dimensional normal distribution if and only if every linear function of X, that is, $Y = \sum_{i=1}^{n} t_i X_i$ is normal.

(9) [**Papoulis Example 7.7**] Let N be a Poisson random variable. Let $X_1, X_2, ...$ be independent and identically distributed Bernoulli random variables $Ber(p)$, that are also independent of N. Let $Y = \sum_{k=1}^{N} X_k$. Find $\mathrm{E}[Y]$ and $\mathrm{var}(Y)$.

(10) [**Papoulis 6-71**] Let X and Y be uniform random variables in the interval $(-1, 1)$ and independent. Find the conditional density of $R = \sqrt{X^2 + Y^2}$ conditioned on $M = \{R \leq 1\}$.

(11) A coin with probability of head p, is tossed repeatedly until k heads occur. Let X be the no. of tosses required. Using conditional expectation (conditioned on the result of the first toss), find $\mathrm{E}X$.

(12) [**Rohatgi 5.2.13**] Let X and Y be independent RVs with PMFs $P\{X = k\} = p_k$, $P\{Y = k\} = q_k$, , $k = 0, 1, 2, ...$, where $p_k, q_k > 0$ and $\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} = 1$. Let

$$P\{X = k | X + Y = t\} = \binom{t}{k} \alpha^k (1-\alpha)^{t-k}, \ \ 0 \leq k \leq t. \tag{0.1}$$

Then prove that

$$p_k = \frac{\exp(-\theta\beta)(\theta\beta)^k}{k!} \text{ and } q_k = \frac{\exp(-\theta)(\theta)^k}{k!} \tag{0.2}$$

where $\beta = \frac{\alpha}{1-\alpha}$ and $\theta > 0$ is arbitrary.

(13) [**Rohatgi 5.2.12**] Let X and Y be independent and identically distributed geometric random variables. Show that $\min(X, Y)$ and $X - Y$ are independent.

(14) A RV X is said to have gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ (also, denoted as $G(\alpha, \beta)$) if

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} \exp(-y/\beta), & 0 < y < \infty, \\ 0, & y \leq 0. \end{cases}$$

   (a) Compute MGF of X.
   (b) Compute $\mathrm{E}X$ and $\mathrm{var}(X)$.

(15) [**Rohatgi Theorem 5.3.4**] Let $X_1, ..., X_n$ be independent RVs such that $X_j \sim G(\alpha_j, \beta)$, $j = 1, ..., n$. Show that $S_n = \sum_{k=1}^{n} X_k$ is Gamma distributed. Also, find the parameters of the distribution.

(16) [**Rohatgi Theorem 5.3.11**] Let $X_1, X_2, ...$ be a sequence of iid RVs having common exponential density with parameter $\beta > 0$. Let $S_n = \sum_{k=1}^{n} X_k$ be the n^{th} partial sum, $n = 1, 2, ...$ and suppose that $t > 0$. If Y = number of

$S_n \in [0, t]$, then show that Y is a Poisson random variable with parameter t/β.

(17) **[Rohatgi 5.3.3]** Let $Y_1 \sim U[0,1], Y_2 \sim U[0, Y_1], ..., Y_n \sim U[0, Y_{n-1}]$. Show that

$$Y_1 \sim X_1,\;\; Y_2 \sim X_1 X_2, ...,\;\; Y_n \sim X_1 X_2 ... X_n, \tag{0.3}$$

where $X_1, X_2, ..., X_n$ are iid $U[0,1]$ RVs. If U is the number of $Y_1, ..., Y_n$ in $[t, 1]$, where $0 < t < 1$, show that U has a Poisson distribution with parameter $-\log(t)$.