E0 232: PROBABILITY AND STATISTICS

PROBLEM SHEET 5
TUTORIAL ON 3^{RD} NOV AND 5^{TH} NOV, 2014 (5.30 PM ONWARDS)
VENUE: CSA-252

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(1) [**Rohatgi 6.2.6, Hajek 2.16**] Let $\theta > 0$ be a constant. Let $X_1, X_2, \ldots$ be iid random variables uniformly distributed on the interval $[0, \theta]$. Let $Y_n = \max\{X_1, \ldots, X_n\}$.
  (a) Show that $Y_n \xrightarrow{P} \theta$. (This means that Y_n converges in probability to a degenerate random variable at θ)
  (b) Does $Y_n \xrightarrow{a.s.} \theta$?
  (c) Does $Y_n \xrightarrow{L^2} \theta$?

(2) [**Hajek 2.8**] Let Θ be uniformly distributed on the interval $[0, 2\pi]$. In which of the three senses ($a.s.$, L^2, P), do each of the following two sequences converge? Identify the limiting random variables, if they exist, and justify your answers.
  (a) $\{X_n\}_{n\geq 1}$ defined by $X_n = \cos(n\Theta)$.
  (b) $\{Y_n\}_{n\geq 1}$ defined by $Y_n = \left|1 - \frac{\Theta}{\pi}\right|^n$.

(3) [**Rohatgi 6.2.8**] Let $\{X_n\}$ be a sequence of RVs such that $P\{|X_n| < k\} = 1$ for all n and some constant $k > 0$. Suppose that $X_n \xrightarrow{P} X$. Show that $X_n \xrightarrow{L^r} X$ for any $r > 0$.

(4) [**Rohatgi 6.2.20b**] Let $\{X_n\}$ be a sequence of RVs such that $P(X_n = e^n) = 1/n^2$, $P(X_n = 0) = 1 - 1/n^2$, and zero otherwise. Does X_n converge in probability? Does X_n converge in r^{th} mean?

(5) [**Rohatgi 6.3.1**] Let $X_1, X_2, \ldots$ be a sequence of iid RVs with common uniform distribution on $[0, 1]$. Also, let $Z_n = \left(\prod_{k=1}^{n} X_k\right)^{1/n}$ be the geometric mean of $X_1, X_2, \ldots, X_n$. Show that $Z_n \xrightarrow{P} c$, where c is a constant. Find c.

These problems have been taken from:

- Chapter 6 of *An Introduction to Probability and Statistics* by Rohatgi and Saleh, second edition.
- Chapter 2 of *An Exploration of Random Processes for Engineers* by Bruce Hajek, 2014.

Some of the problems have been modified slightly. The problems given here are not straightforward, and hence, solving other problems from suggested textbooks before trying these may be useful. Solutions of these problems will be discussed during the tutorial session.

(6) **[Rohatgi 6.3.4,6.3.6]** Let $X_1, X_2, \ldots$ be a sequence of RVs. Prove the law of large numbers when:

(a) $Var(X_k) < \infty$ for all k and $\frac{1}{n^2} \sum_{k=1}^{n} Var(X_k) \to 0$ as $n \to \infty$.

(b) $Var(X_k) \leq C$ for all k and $Cov(X_i, X_j) \to 0$ as $|i - j| \to \infty$.

(7) A random quantity is measured multiple times. Let E_n be the error in measurement at the n^{th} trial. It is given that for $n > N$

$$P(E_n > \epsilon) \leq n^{1-\alpha} \qquad \text{for some } \alpha > 0.$$

(a) For what values of α can we say that with probability 1, the error is at most ϵ infinitely often?

(b) Can we say, under some conditions, that with probability 1, the error is at least ϵ infinitely often?

(8) **[Hajek 2.30]** Suppose that you are given one unit of money (for example, a million dollars). Each day you bet a fraction α of it on a coin toss (assume coin is unbiased). If you win, you get double your money back, whereas if you lose, you get half of your money back. Let W_n denote the wealth you have accumulated (or have left) after n days. Identify in what sense(s), W_n converges, and when it does, identify the limiting random variable

(a) for $\alpha = 0$ (pure banking),

(b) for $\alpha = 1$ (pure betting),

(c) for general α.

(d) What value of α maximizes the expected wealth $E[W_n]$? Would you recommend using that value of α?

(e) What value of α maximizes the long term growth rate of W_n?
(Hint: Consider $\log W_n$ and apply the LLN.)

Possible Solutions

NOTE: These are possible solutions for the above problems. They may not be correct, and even if they are correct, they may not be the best way to solve the problems.

(1) Done in class

(2) **Part (a)**

Suppose there is a rv X such that $\cos n\Theta \xrightarrow{L^2} X$. Then $E[(\cos n\Theta - X)^2] \to 0$. Thus, for m, n large enough, we can write

$$E[(\cos m\Theta - \cos n\Theta)^2] = E[\{(\cos m\Theta - X) + (X - \cos n\Theta)\}^2]$$
$$= E[(\cos m\Theta - X)^2] + E[(X - \cos n\Theta)^2] + 2E[(\cos m\Theta - X)(X - \cos n\Theta)]$$

The first two terms go to zero for large m, n and we can bound third term by Cauchy-Scwarz inequality to show

$$\begin{aligned} E[(\cos m\Theta - X)(X - \cos n\Theta)] &\le E[|(\cos m\Theta - X)(X - \cos n\Theta)|] \\ &\le \sqrt{E[(\cos m\Theta - X)^2]E[(X - \cos n\Theta)^2]} \end{aligned}$$

which also goes to zero. Thus if $\cos n\Theta \xrightarrow{L^2} X$, then $E[(\cos m\Theta - \cos n\Theta)^2] \to 0$ as $m, n \to \infty$. On the other hand, we can compute

$$E[(\cos m\Theta - \cos n\Theta)^2] = \frac{1}{2\pi}\int_0^{2\pi} (\cos^2 m\theta + \cos^2 n\theta - 2\cos m\theta \cos n\theta) d\theta$$
$$\begin{cases} 0 & \text{if } m = n \\ 1/2 & \text{if } m \neq n \end{cases}$$

Thus, $E[(\cos m\Theta - \cos n\Theta)^2]$ does not always go to zero for large m, n. This leads to a contradiction. Hence, $\cos n\Theta$ does not converge in L^2.

Observe that $P(|\cos n\Theta| \le 1) = 1$. If $\cos n\Theta \xrightarrow{P} X$, then we can use Problem (3) to claim $\cos n\Theta \xrightarrow{L^2} X$. Since, the latter is not true, hence $\cos n\Theta$ does not converge in probability.

Similarly, if $\cos n\Theta \xrightarrow{a.s.} X$, we can say $\cos n\Theta \xrightarrow{P} X$. Since, the latter is not true, $\cos n\Theta$ does not converge in almost sure sense.

Part (b)

Note for any $\theta \in (0, 2\pi)$, $|1 - \frac{\theta}{2\pi}| < 1$. Hence,

$$Y_n(\omega) = |1 - \frac{\theta}{2\pi}|^n \to 0$$

Thus,

$$P(\{\omega : Y_n(\omega) \to 0\}) = P(0, 2\pi) = 1.$$

Hence, $Y_n \xrightarrow{a.s.} 0$, which also implies $Y_n \xrightarrow{P} 0$. Similarly, one can compute

$$E[|Y_n - 0|^2] = \frac{2}{2n+1} \to 0 \qquad \text{as } n \to \infty.$$

Hence, $Y_n \xrightarrow{L^2} 0$.

(3) Since $P(|X_n| < k) = 1$ and $P(|X_n - X| > \epsilon) \to 0$ for all $\epsilon > 0$,

$$\begin{aligned} P(|X| > k + \epsilon) &= P(|X_n + X - X_n| > k + \epsilon) \\ &\le P(|X_n| + |X_n - X| > k + \epsilon) \\ &\le P(|X_n| > k) + P(|X_n - X| > \epsilon) \to 0. \end{aligned}$$

So $P(|X| \le k + \epsilon) = 1$ for all $\epsilon > 0$, and hence, $P(|X| \le k) = 1$. Thus, for any n,

$$\begin{aligned} P(|X - X_n| > 2k) &\le P(|X_n| + |X| > 2k) \\ &\le P(|X_n| > k) + P(|X| > k) = 0. \end{aligned}$$

So $P(|X - X_n| \le 2k) = 1$.

We now use the following fact from [Rohatgi, page 75]: If Z is a non-negative rv, then $EZ = \int\limits_0^\infty (1 - F_Z(z)dz$.

Now define $Z_n = |X_n - X|^r$. Then Z_n is non-negative random variable with $P(Z_n \le (2k)^r) = 1$, and $P(Z_n > \epsilon) \to 0$ for any $\epsilon > 0$. So

$$\begin{aligned} \int_0^\infty (1 - F_{Z_n}(z)dz &= \int_0^{(2k)^r} P(Z_n > z)dz \\ &= \int_0^\epsilon P(Z_n > z)dz + \int_\epsilon^{(2k)^r} P(Z_n > z)dz \\ &\le \int_0^\epsilon dz + \int_\epsilon^{(2k)^r} P(Z_n > \epsilon)dz \\ &= \epsilon + P(Z_n > \epsilon)((2k)^r - \epsilon) \\ &\to \epsilon \qquad \text{as } P(Z_n > \epsilon) \to 0. \end{aligned}$$

So, $\lim_{n\to\infty} E[|X_n - X|^r] \le \epsilon$ for all $\epsilon > 0$. So $\lim_{n\to\infty} E[|X_n - X|^r] = 0$.

(4) For any $\epsilon > 0$,

$$P(|X_n - 0| > \epsilon) \le P(X_n \ne 0) = \frac{1}{n^2} \to 0.$$

Hence, $X_n \xrightarrow{P} 0$.

Recall if $X_n \xrightarrow{L^2} X$ for some r.v. X, then $X_n \xrightarrow{P} X$. Since, we already know that $X_n \xrightarrow{P} 0$, it suffices to check whether $X_n \xrightarrow{L^2} 0$ or not. Observe that

$$E[|X_n - 0|^r] = \frac{e^{rn}}{n^2} \to \infty \qquad \text{as } n \to \infty.$$

Hence, X_n does not converge in L^r.

(5) We know if X_n is uniform on $[0,1]$, then $-\log X_n$ has exponential distribution with $\lambda = 1$. Hence, $E[-\log X_n] = 1$ and $Var(-\log X_n) = 1 < \infty$ for all n. Now, check that

$$-\log Z_n = -\log \left(\prod_{k=1}^{n} X_k \right)^{1/n} = \frac{1}{n} \sum_{k=1}^{n} (-\log X_k).$$

Hence, by law of large numbers, $(-\log Z_n) \xrightarrow{P} 1$. So, for any $\delta > 0$, we have $P(|-\log Z_n - 1| \geq \delta) \to 0$, or we can write

$$\begin{aligned} &P(\{eZ_n \leq e^{-\delta}\} \cup \{eZ_n \leq e^{\delta}\}) \\ &= P(\{\log(eZ_n) \leq -\delta\} \cup \{\log(eZ_n) \leq \delta\}) \\ &= P(|\log(eZ_n)| \geq \delta) = P(|\log Z_n + 1| \geq \delta) \to 0. \end{aligned}$$

Thus, for any $\epsilon > 0$,

$$\begin{aligned} P(|Z_n - e^{-1}| \geq \epsilon) &= P(|eZ_n - 1| \geq \epsilon) \\ &= P(\{eZ_n \leq 1 - e\epsilon\} \cup \{eZ_n \geq 1 + e\epsilon\}) \\ &\leq P(\{eZ_n \leq (1 + e\epsilon)^{-1}\} \cup \{eZ_n \geq 1 + e\epsilon\}) \end{aligned}$$

Since, $(1+a)^{-1} > (1-a)$ for any $a > 0$. Putting $\delta = \log(1 + e\epsilon)$ above and combining with previous convergence, we get $Z_n \xrightarrow{P} e^{-1}$.

(6) **Part (a)**

It is given that $\frac{1}{n^2} \sum_{k=1}^{n} Var(X_k) \to 0$ as $n \to \infty$.

$$E\left[\left(\frac{S_n}{n} - \mu \right)^2 \right] = \frac{1}{n^2} Var(S_n) = \frac{1}{n^2} \sum_{k=1}^{n} Var(X_k) \to 0.$$

So, $\frac{S_n}{n} \xrightarrow{L^2} \mu$, and hence, $\frac{S_n}{n} \xrightarrow{P} \mu$, which proves LLN.

Part (b)

It is given $Cov(X_i, X_j) \to 0$ as $|i - j| \to \infty$. This means given any $\epsilon > 0$, there exists an integer i_ϵ such that for all $i > i_\epsilon$, $Cov(X_k, X_{k+i}) < \epsilon$. We also know that $Var(X_k) \leq C$ for all k, and by Cauchy-Schwarz inequality

$Cov(X_k, X_l) \le \sqrt{Var(X_k)Var(X_l)} \le C$ for all k, l. Now,

$$
\begin{aligned}
&E\left[\left(\frac{S_n}{n} - \mu\right)^2\right] = \frac{1}{n^2} E\left[\left(\sum_{k=1}^{n} X_k - \mu\right)^2\right] \\
&= \frac{1}{n^2}\sum_{k=1}^{n} Var(X_k) + \frac{2}{n^2}\sum_{k=1}^{n}\sum_{l=k+1}^{n} Cov(X_k, X_l) \\
&= \frac{1}{n^2}\sum_{k=1}^{n} Var(X_k) + \frac{2}{n^2}\sum_{i=1}^{n-1}\sum_{k=1}^{n-i} Cov(X_k, X_{k+i}) \\
&= \frac{1}{n^2}\sum_{k=1}^{n} Var(X_k) + \frac{2}{n^2}\sum_{i=1}^{i_\epsilon}\sum_{k=1}^{n-i} Cov(X_k, X_{k+i}) + \frac{2}{n^2}\sum_{i=i_\epsilon+1}^{n}\sum_{k=1}^{n-i} Cov(X_k, X_{k+i}) \\
&< \frac{1}{n^2}\sum_{k=1}^{n} C + \frac{2}{n^2}\sum_{i=1}^{i_\epsilon}\sum_{k=1}^{n-i} C + \frac{2}{n^2}\sum_{i=i_\epsilon+1}^{n}\sum_{k=1}^{n-i} \epsilon \\
&\le \frac{nC}{n^2} + \frac{2i_\epsilon nC}{n^2} + \frac{2n^2\epsilon}{n^2}.
\end{aligned}
$$

So, for any $\epsilon > 0$,

$$\lim_{n\to\infty} E\left[\left(\frac{S_n}{n} - \mu\right)^2\right] < 2\epsilon,$$

which means the limit is 0. So, $\frac{S_n}{n} \xrightarrow{L^2} \mu$, and hence, $\frac{S_n}{n} \xrightarrow{P} \mu$, which proves LLN.

(7) Fact: $\sum_n \frac{1}{n^p} < \infty$ for any $p > 1$, and $\sum_n \frac{1}{n^p} = \infty$ for any $p \le 1$. At least, you should remember $\sum_n \frac{1}{n^2} < \infty$ and $\sum_n \frac{1}{n} = \infty$.

Using above fact, if $\alpha > 2$, then $\sum_n \frac{1}{n^{\alpha-1}} < \infty$. So,

$$
\begin{aligned}
\sum_{n=1}^{\infty} P(E_n > \epsilon) &= \sum_{n=1}^{N} P(E_n > \epsilon) + \sum_{n=N+1}^{\infty} P(E_n > \epsilon) \\
&\le N + \sum_{n=N+1}^{\infty} \frac{1}{n^{\alpha-1}} < \infty.
\end{aligned}
$$

Hence, by first part of Borel-Cantelli lemma, $P\left(\bigcap_n \bigcup_{k\geq n} \{E_n > \epsilon\}\right) = 0$. So, for $\alpha > 2$,

$$
\begin{aligned}
P(\{E_n \leq \epsilon\}\ i.o.) &= P\left(\bigcap_n \bigcup_{k\geq n} \{E_n \leq \epsilon\}\right) \\
&\geq P\left(\bigcup_n \bigcap_{k\geq n} \{E_n \leq \epsilon\}\right) \\
&= 1 - P\left(\left\{\bigcup_n \bigcap_{k\geq n} \{E_n \leq \epsilon\}\right\}^c\right) \\
&= 1 - P\left(\bigcap_n \bigcup_{k\geq n} \{E_n > \epsilon\}\right) = 1.
\end{aligned}
$$

To show $P(\{E_n \geq \epsilon\}\ i.o.) = P\left(\bigcap_n \bigcup_{k\geq n} \{E_n \geq \epsilon\}\right) = 1$, we can apply second part of Borel-Cantelli lemma, if E_n are independent random variables and $\sum_n P(E_n > \epsilon) = \infty$. To comment whether we obtain an infinite sum, we need a lower bound on $P(E_n > \epsilon)$ whose sum over all n is infinite.

(8) Try it yourself.