E0 232: PROBABILITY AND STATISTICS

PROBLEM SHEET 2
TUTORIAL ON 17^{TH} SEPTEMBER, 2016
VENUE: CSA - 117 (11.30 AM ONWARDS)

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(1) [**Rohatgi 2.2.1, 2.3.1**] Let X be the number of heads in three tosses of a coin. What is Ω? What are the values that X assigns to points of Ω? What are the events $\{X \leq 2.75\}$, $\{0.5 \leq X \leq 1.72\}$? Write the distribution function of X assuming that p is probability of getting head.

(2) [**Rohatgi 2.2.3**] Let X be an RV (random variable) on the sample space $(\Omega, \mathcal{F})$. Is $|X|$ also an RV? If X is an RV that takes only nonnegative values, is $\sqrt{X}$ also an RV?
Hint: Use the definition of RV in the following form. For any real x, the RV X satisfies $X^{-1}\big((-\infty, x]\big) \in \mathcal{F}$. Using this and the properties of σ-field, show that the same holds for $|X|$ and $\sqrt{X}$.

(3) [**Rohatgi 2.3.3,2.4.2,2.4.9; Ross 2.9; Hajek 1.12**] Are the following functions probability distribution functions? If so, find the corresponding probability density or mass functions.

(a) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } 0 \leq x < 1 \\ 3/5 & \text{if } 1 \leq x < 2 \\ 4/5 & \text{if } 2 \leq x < 3 \\ 9/10 & \text{if } 3 \leq x < 3.5 \\ 1 & \text{if } x \geq 3.5 \end{cases}$

(b) $F(x) = \frac{1}{\pi} \tan^{-1} x$ for $-\infty < x < \infty$

(c) $F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 - \frac{1}{x} & \text{if } x > 1 \end{cases}$

(d) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$ $\quad (\lambda > 0$ is a constant$)$

These problems have been taken from:

- Chapter 2 of *An Introduction to Probability and Statistics* by Rohatgi and Saleh, 2nd ed.
- Chapter 2 of *Introduction to Probability Model* by Sheldon Ross, tenth edition.
- Chapter 1 of *An Exploration of Random Processes for Engineers* by Bruce Hajek, 2014.

Some of the problems have been modified slightly. The problems given here are not straightforward, and hence, solving other problems from suggested textbooks before trying these may be useful. Solutions of these problems will be discussed during the tutorial session.

(e) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (1+x)e^{-x} & \text{if } x \geq 0 \end{cases}$

(f) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } x \geq \frac{1}{2} \end{cases}$

(g) $F(x) = \begin{cases} \frac{e^{-x^2}}{4} & \text{if } x < 0 \\ 1 - \frac{e^{-x^2}}{4} & \text{if } x \geq 0 \end{cases}$

(h) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 + e^{-x} & \text{if } 0 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$

(i) $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 + \frac{x}{20} & \text{if } 0 \leq x < 10 \\ 1 & \text{if } x \geq 10 \end{cases}$

(j) $F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{(x-2)^2}{8} & \text{if } 1 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$

(4) [**Ross 2.33,2.34; Rohatgi 2.4.5,2.4.6**] For what values of K do the following functions define the probability mass function or density function of some RV? Also find the corresponding distribution function. In each case, assume $f(x) = 0$ to be zero for any value or range of x not mentioned below.

   (a) $f(x) = K/N$ for $x = 1, 2, ..., N$

   (b) $f(x) = K\frac{\lambda^x}{x!}$ for $x = 0, 1, 2, ...$

   (c) $f(x) = Ke^{-|x|}$ for $-\infty < x < \infty$

   (d) $f(x) = K(1 - x^2)$ for $-1 < x < 1$

   (e) $f(x) = K(4x - 2x^2)$ for $0 < x < 2$

(5) [**Ross 2.12**] On a multiple-choice exam with three possible answers for each of the five questions, what is the probability that a student would get four or more correct answers just by guessing?

(6) [**Ross 2.23**] A coin has probability p of coming up heads.

   (a) Suppose the coin be flipped n times, and Y be the number of heads. Show that Y will have a binomial distribution, given by

$$P(\{Y = r\}) = \binom{n}{r} p^r (1-p)^{n-r}, \qquad r = 0, 1, ..., n.$$

(b) It is successively flipped until the r^{th} head appears. Show that the number of flips required, X, will have a negative binomial distribution, given by

$$P(\{X = n\}) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \qquad n \geq r.$$

Hint: How many successes must there be in the first $n-1$ trials?

(7) [**Ross 2.28**] Suppose that we want to generate a random variable X that is equally likely to be either 0 or 1, and that all we have at our disposal is a biased coin that, when flipped, lands on heads with some (unknown) probability p. Consider the following procedure:
- Flip the coin, and let O_1, either heads or tails, be the result.
- Flip the coin again, and let O_2 be the result.
- If O_1 and O_2 are the same, return to the first step.
- If O_2 is heads, set $X = 0$, otherwise set $X = 1$.

(a) Show that the random variable X generated by this procedure is equally likely to be either 0 or 1. (b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different, and then sets $X = 0$ if the final flip is a head, and sets $X = 1$ if it is a tail?

(8) [**Ross 2.30**] A Poisson random variable X takes only non-negative integer values $0, 1, 2, \ldots$ and its probability mass function is given by $P\{X = i\} = \frac{\lambda^i}{i!} e^{-\lambda}$ for $i = 0, 1, 2, ..,$ where $\lambda > 0$ is a parameter. Show that $P\{X = i\}$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is largest integer not exceeding λ.
Hint: Compute $P\{X = i\}/P\{X = i-1\}$ for each i.

(9) [**Hajek 1.17**] Let X be exponentially distributed with parameter $\lambda > 0$. Find the distribution functions for the random variables $Y = e^X$ and $Z = \min(X, 3)$.

(10) The distribution function $F(x) = P(X \leq x)$ satisfies certain properties. Can you derive similar properties for the function $G(x) = P(X \geq x)$?

(11) In a 10-over cricket match, the runs that can be scored by a poor batsman is given by a Poisson distribution (described above) with parameter $\lambda = 10$. On the other hand, the runs that a good batsman can score is given by a Poisson distribution with parameter $\lambda = 30$. If a batsman scores 20 runs in the match, would you judge him as good or poor?